Journal of Statistical Physics, Vol. 48, Nos. 5/6, 1987

Fields as Kolmogorov Flows

B. Misra[1]

It is widely recognized that for highly unstable dynamical systems there exists a fundamental limitation on predictability and determinism. An important class of such highly unstable systems is the class of K-flow, which is further characterized by the existence of time-asymmetric objects in the form of K-partition. Our recent approach to the problem of irreversibility has shown that when the implications of the limitation on determinism arising from strong form of instability and those of the existence of K-partition are consistently taken into account, one is naturally led from the physically unrealizable deterministic evolution of phase points to an entropy-increasing stochastic Markovian evolution. Furthermore, this transition is not the result of extraneously imposed coarse graining and/or approximation schemes, but can be brought about by an invertible transformation whose existence and construction are determined by the nature of the instability of the dynamical system itself. After a brief review of this theory which also contains some relatively new remarks, we prove that classical Klein–Gordon field (both massive and massless) possess the structure of K-flow. This seems to provide the first examples of relativistic systems that are K-flows. Some of the implications of this result are briefly discussed. From a mathematical point of view, this seems to be a first step toward an ergodic theory of partial differential equations. In the process, we also provide an independant group-theoretic proof of the existence of incoming and outgoing subspaces of the scattering theory of Lax and Phillips for the wave equation.

KEY WORDS: K-flows; Λ-transformation; wave equation incoming-outgoing subspaces.

1. INTRODUCTION

The broken time symmetry implied by the laws of monotonic increase of entropy appears to be of far more fundamental significance in physical theories than previously realized. It is an understatement to say that Prof.

[1] Instituts Internationaux de Physique et de Chimie, Université Libre de Bruxelles, B 1050 Bruxelles, Belgium, and School of Physical Sciences, Jawaharlal Nehru University, New Delhi, India.

822/48/5-6-22

I. Prigogine has taken a leading role over a period of more than three decades in the quest of the dynamical origin of this broken time symmetry. Not only do his own ideas and work mark significant turning points in this search, but also his characteristically insightful and optimistic appreciation of the work of others have encouraged many to pursue this continuing quest. Having benefitted both from his inspiring ideas and his constant, unfailing encouragement, it is with a feeling of joy and honor that I dedicate this essay to him to celebrate his 70th birthday.

As a result of advances in modern ergodic theory and the theory of dynamical systems, it is now widely recognized that, for systems (such as K-flows) whose motion exhibits a high degree of *instability* (i.e., sensitive and irregularly discontinuous dependence on initial conditions), there is a *fundamental limitation* on the deterministic description of time evolution in terms of motion of phase points along phase space trajectories. This limitation is, in fact, as fundamental (but, of course, of distinct physical origin) as the limitation on physical realizability of phase space trajectories arising from the quantum uncertainty principle. And this limitation can occur even for macroscopic systems for which the limitation arising from quantum principles is negligible.

These points are, indeed, so well recognized today that they hardly need further elaboration.[2] What is, however, not sufficiently emphasized is that, since (for systems exhibiting a high degree of instability) the limitation on the physical realizability of deterministic evolution along phase space trajectories is fundamental in character, the concept of the deterministic motion of phase points should be eliminated from the theoretical description of time evolution of such systems.

It is just this point (namely, the elimination of deterministic motion of individual phase points from the theory) that forms the starting point of our recent approach to the problem of broken time symmetry.[3–15] This elimination cannot, however, be achieved simply by replacing the motion of individual phase points by the Liouvillian evolution of Gibbs distribution functions, for the simple reason that the Liouville equation is obtained on the basis of the deterministic motion of individual phase points that we are attempting to eliminate from the theory! (Moreover, the

[2] See, e.g., Lighthill,[1] especially p. 47, where he writes, "A fundamental uncertainty about the future is there, indeed, even on the supposedly solid basis of the good old laws of motion of Newton." Although the recent upsurge of interest in chaotic dynamics has brought to light many and often unexpectedly simple systems whose motions are highly unstable and has thus furthered the wider recognition of the fundamental limitations to predictibility and determinism arising from the high degree of instability of motion, this limitation had been recognized much earlier. See, e.g., Feynman,[2] who stressed that in Newtonian theory, if the initial conditions of a complex system are known to a certain accuracy, then all the accuracy is lost in less time than it takes to state that accuracy in words!

Liouvillian evolution does not, of course, break the symmetry between the two directions of time.)

At this point, the conventional approach to the problem of irreversibility introduces extraneously some form of coarse-graining or contraction of description and certain approximation schemes (such as the weak coupling limit, Boltzmann–Grad limit, etc.) in order to arrive at an evolution equation with broken time symmetry.[16]

The viewpoint of our approach is, however, radically different from that of the conventional approach. For, we seek, not to explain the broken time symmetry as arising from approximations or coarse-granning introduced *extraneously* into the dynamics, but to introduce the broken time symmetry as a fundamental physical principle and to investigate fundamental implications of this principle.[10]

To achieve the desired elimination of deterministic motion of phase points from the theory of time evolution of highly unstable systems, and to arrive at a theory of broken time symmetry, we have to take into account another important feature of instability characteristic of the K-flows. Roughly speaking, a K-flow is characterized by the existence of a nontrivial partition (called K-partition) of the phase space, which has the property that under dynamical evolution the partition becomes *progressively* finer as time increases (toward the future), with each cell of the partition contracting asymptotically to a point for $t \to \infty$. On the other hand, if the dynamical evolution is extrapolated toward the past, the K-partition becomes progressively coarser as we extrapolate toward more and more remote past.

The property of the K-partition just mentioned shows that with increasing time it should become increasingly more difficult to distinguish the points belonging to the *same* cell of the K-partition, since they come closer to each other and merge into each other asymptotically for $t \to +\infty$.

It is remarkable that when the existence of an intrinsically time-asymmetric object, the K-partition, and the increasing degree of difficulty of distinguishing the points in the same cell as time increases (in the forward direction) are consistently taken into account, one is naturally led from the deterministic evolution of the system to an entropy-encreasing probabilistic Markovian evolution. What is noteworthy about this passage from deterministic evolution to entropy-increasing stochastic evolution is that it is not the result of extraneously imposed coarse-graining or approximation schemes, but can be brought about by an *invertible* transformation whose existence and construction are determined by the intrinsic nature of the instability of the dynamical system.

Somewhat more precisely, let S_t denote the group of transformations describing the deterministic motion of phase points ω of a K-flow and let

U_t be the unitary group induced from S_t that describes the Liouvillian evolution of Gibbs density functions ρ:

$$(U_t \rho)(\omega) = \rho(S_{-t}\omega)$$

Then, it can be proved that there exists a bounded transformation Λ having the following properties: (i) If ρ is a (Gibbs) probability distribution, then so is $\Lambda\rho$; and (ii) it satisfies the inertwining relation $\Lambda U_t = W_t \Lambda$, with a semigroup W_t, which describes the evolution of Gibbs distribution function under a Markov process for $t \geqslant 0$ and such that the (negative) entropy $\int (W_t \rho) \log(W_t \rho)\, d\mu$ is a *monotonically decreasing* function of time. Moreover, Λ can be chosen so that it has an unbounded (but densly defined) inverse Λ^{-1}, in which case W_t is nonunitarily similar to U_t: $W_t = \Lambda U_t \Lambda^{-1}$ for $t \geqslant 0$.

In explanation of property (ii), it may added that W_t is such that $P(t, \omega, \Delta)$ defined by $P(t, \omega, \Delta) = (W_t^* \varphi_\Delta)(\omega)$ satisfies all the properties (including the Chapman–Kolmogorov equation) required of the transition probability (from the phase point ω to the region Δ in time $t \geqslant 0$) of a genuinely stochastic Markov process. Here W_t^* denotes the Hermitian conjugate of W_t and φ_Δ denotes the characteristic function of the region Δ. For more details the reader should consult the work cited earlier, especially Refs. 4–8.

It is not known if the K-flow property is also *necessary* for the existence of such an invertible transformation Λ. It is known, however, that the mixing property is necessary and that Λ does *not* exist for a more stable dynamical evolution than the mixing systems.

The underlying viewpoint of this result is that *not all* (Gibbs) probability distributions ρ can be physically prepared, but only a proper subset of them that are of the form $\Lambda\rho$ are physically realizable and the probabilitistic evolution W_t (for $t \geqslant 0$) that maps this class of physically realizable states into itself describes the *physical evolution* of the K-flows. In this approach, then, the origin of the monotonic increase of entropy is the instability of motion of the type exhibited by the K-flow, which on one hand renders the deterministic description of motion to be an unphysical idealization and in addition implies a limitation on physically realizable states.

Let us mention here that this result on the existence of invertible Λ for K-flows and its nonexistence for stable evolutions establishes rigorously the mathematical and physical consistency of an important idea, first discussed by Prigogine *et al.*[17] that the time-reversible dynamical evolution and entropy-increasing semigroup evolution of physical systems could be related through a nonunitary similarity transformation.

Since it is still widely supposed that repeated collisions are essential for entropy increase, it is worthwhile to mention that according to the theory sketched here, entropy increase can occur even in the absence of collisions. This is because, as stated before, the origin of entropy increase is the form of instability that is characteristic of K-flows, and such instability can occur even in the absence of collision; as, for example, in the case of geodesic flow in compact Riemannian space of constant negative curvature. In fact, it can be shown that if the spatial curvature of the universe is negative and its spatial hypersurface compact, then the free (geodesic) motion of intergalactic gas is entropy-increasing, even though molecular collisions are extremely rare due to the extremely low density of intergalactic gas.[18] It may be mentioned in this connection that the possibility of entropy increase in the absence of collisions has been recently considered by Lighthill,[1] who cites the phenomenon of the so-called "bow shock wave" in the solar wind as a possible example of such a collisionless entropy-increasing phenomenon.

In view of the important role of the K-flow property in the theory of irreversibility, it is of obvious importance to study systems of physical interest that exhibit this property. Many physically interesting systems, such as the hard-sphere gas in a box, the Lorentz gas model,[19–21] the infinite harmonic lattices,[22–24] and the previously mentioned example of geodesic flow in compact space of negative curvature,[25,26] are known to be K-flows. The main purpose of this article is to show that relativistic (free) fields also belong to this class. (Actually, I shall consider only the case of massless and massive Klein–Gordon field. A more detailed development of this work will be presented in a forthcoming publication in collaboration with I. Antoniou, where the case of electromagnetic field as well as the field of gravitional waves, etc., will also be considered.)

In the concluding section, I briefly discuss some of the implications of the result. Here, I note that this result implies that the assumption that Cauchy data can be arbitrarily specified and that their deterministic time evolution in accordance with the field equation can be followed constitutes an unphysical idealization. Indeed, even independent of this result, the problematics of the operational meaning of specifying Cauchy data (on a spatial hypersurface) is far more complicated that the problematics of specifying initial conditions of classical systems with finite degrees of freedom. I briefly return to this point in the concluding section.

This work also seems to provide the first examples of relativistic systems that are K-flows. Therefore, it can serve as the basis of studying the relativistic transformation properties of irreversible semigroup evolution law as well as of the related *internal time operator*. Finally, from a mathematical point of view, this work seems to be a first step toward an ergodic theory of partial differential equations.[28]

2. *K*-FLOWS AND INTERNAL TIME OPERATOR

In order to prove the K-flow property of Klein–Gordon field, we shall need a characterization of K-flow[27,28] in terms of the existence of the so-called *internal time operator* given below. To formulate this criterion, we shall introduce the notation $(\Omega, \mathscr{B}, \mu, S_t)$ to denote an (abstract) dynamical system. Here $(\Omega, \mathscr{B}, \mu)$ denotes a probability measure space and S_t denotes a group of (measurable) transformation of Ω onto itself under which the measure μ is invariant.

Physically, Ω represents (a constant energy surface of) the phase space, S_t represents the group of dynamical motion, and μ is invariant (Liouville) measure, which is defined on a suitable σ-algebra $\mathscr{B}$ of subsets of Ω, the measurable subsets. For convenience, μ is normalized so that $\mu(\Omega) = 1$.

Let $P_{-\infty}$ denote the one-dimensional projection in $\mathscr{L}^2_\mu$ on to the unit function 1 (the microcanonical ensemble) and $\mathscr{H}$ the subspace of $\mathscr{L}^2_\mu$ that is orthogonal to the unit function 1. The desired criterion can now be formulated as follows.

Theorem. In order that the dynamical system $(\Omega, \mathscr{B}, \mu, S_t)$ be a K-flow, it both necessary and sufficient that there exists a self-adjoint operator T (called the internal time operator) having the following properties:

(a) $U_t^* T U_t = T + tI$ on $\mathscr{H}$, where U_t denotes the unitary group induced from S_t:

$$(U_t f)(\omega) = f(S_{-t}\omega)$$

(b) If F_λ denotes the spectral projections of T, i.e., $T = \int_{-\infty}^{+\infty} \lambda \, dF_\lambda$, then the projections $P_\lambda = F_\lambda + P_{-\infty}$ map the class of Gibbs probability distributions into itself, i.e., P_λ preserve positivity and probability normalization of functions.

We need not stop here for a detailed proof of this result, which was stated in Ref. 18. Let us only mention that the necessity of the existence of such a T for K-flows follows from the considerations of Ref. 3. Condition (b) was noted stated explicitly in that work, but it follows from the fact that P_λ is constructed to be the projection of *conditional expectation* with respect to the partition $S_\lambda \xi_0$ (or the associated σ-algebra), where ξ_0 is the K-partition (see Lemma 2[7]). The sufficiency, on the other hand, follows from a rephrasing of the considerations in Ref. 13. In fact, if such an operator T exists, then the family of projections P_λ satisfies, in addition to condition (b), the following properties:

(i) $P_\lambda > P_\mu$ if $\lambda > \mu$.

(ii) $\lim_{\lambda \to +\infty} P_\lambda = I$ and $\lim_{\lambda \to -\infty} P_\lambda = P_{-\infty}$.

(iii) $U_t P_\lambda U_t^* = P_{\lambda + t}$.

[Condition (iii) is equivalent to condition (a) on T and the fact that $P_{-\infty}$ commutes with U_t.]

Now, condition (b) implies that the projections P_λ are the projections of conditional expectation with respect to partitions ξ_λ of Ω,[29] and condition (iii) in conjunction with (i) entails that $S_t \xi_\lambda = \xi_{\lambda+t}$ and $\xi_{\lambda+t}$ is finer than ξ_λ (for $t > 0$). Conditions (ii) ($\lim_{\lambda \to +\infty} P_\lambda = I$) imply that the coarsest partition that is finer than all ξ_λ is the partition of the phase space into points and similarly the condition $P_\lambda \to P_{-\infty}$ as $t \to -\infty$ implies that the finest partition that is coarser than all partitions ξ_λ is the trivial partition whose cells are of either measure 0 or 1. Thus, the partition ξ_0 corresponding to the projection P_0 has all the properties of a K-partition and the dynamical system is a K-flow.

A family of projections P_λ satisfying condition (b) and conditions (i)–(iii) may be called a *system of imprimitivity of conditional expectations* for the dynamical group U_t. The above theorem may be rephrased to say that the K-flow property of a dynamical system is equivalent to the existence of a system of imprimitivity of conditional expectations with respect to the group of the dynamical motion.

As a digression, I note that this rephrasing is useful because it allows one to formulate a quantum analogue of K-flow instability in terms of the existence of a system of imprimitivity of (*noncommutative*) *conditional expectation* (on the algebra of observables) for the group of automorphisms describing the Heisenberg evolution of observables. If such an imprimitivity system of noncommutative conditional expectations exists, then one can also define a form of K-partition of the phase space (which in the quantum case would be the space of all pure states or indecomposable positive linear functionals on the algebra of observables) such that under dynamical evolution the points on the same cell approach each other with increasing time in the sense that all the points on the cell assign the same expectation value to each of the operators belonging to progressively growing subalgebras of the algebra of all observables. The time-inverted states of a cell of the K-partition, on the other hand, grow progressively more apart from each other with time progressing toward the future. Thus, the existence of an imprimitivity system of (noncommutative) conditional expectations for the Heisenberg evolution of operators does express a form of instability of the dynamical evolution of operators. Since my aim here is to prove the K-flow property of fields considered as *classical* systems, I shall not elaborate further on this, but only refer to Ref. 15. There some steps toward developing this idea are taken and a theory of the quantum measurement process is given that avoids the well-known difficulties of earlier theories, especially Bell's reversibility argument.[30]

Before closing this section, I note the construction of Λ in terms of the internal time operator T. This will permit us to show how the transformation Λ and the resulting entropy-increasing evolution W_t consistency take into account the progressively increasing difficulty of distinguishing the points of a given cell of the K-partition as the cell contracts under the dynamical evolution with increasing time, whereas the physically unrealizable deterministic evolution not only fails to do so, but does the opposite. The transformation can actually be constructed as a suitable decreasing function of the internal time operator T. More precisely,

$$\Lambda = \int h(\lambda)\, dF_\lambda + P_{-\infty}$$

where $h(\lambda)$ is a suitable function monotonically decreasing to 0 for $\lambda \to +\infty$. For the precise conditions on $h(\lambda)$, see Ref. 7. It will suffice to say here that a possible choice of $h(\lambda)$ (for K-flows of finite Kolmogorov–Sinai entropy), for which some physical motivations can be advanced, is

$$h(\lambda) = \frac{1}{1 + e^{K\lambda}}$$

Here K denotes the Kolmogorov–Sinai (KS) entropy of the system. This is the function adopted in defining Λ in the considerations given below.

A precise definition of KS entropy need not be given here.[27,28] I only mention that the KS entropy is (at least under some smoothness hypothesis about the flow) the phase space average of Liapounov exponents characterizing the (exponential) divergence of the phase space trajectories of neighboring initial conditions. Physically, it thus serves as a measure of the limitation to the deterministic description of time evolution. The KS entropy can also be related to rate of entropy production.[31]

Let us now see how the evolution under $W_t = \Lambda U_t \Lambda^{-1}$ of Gibbs distribution functions $\Lambda\rho$ takes into account the *progressively increasing* difficulty of distinguishing the points in the same cell of the K-partitions as it contracts with increasing time, while the deterministic dynamical evolution fails to do so. As mentioned earlier, the projection P_λ is the projection of conditional expectation with respect to the partition $\xi_\lambda = S_\lambda \xi_0$ into which the K-partition evolves in time λ. For a given Gibbs density function ρ, the part $P_\lambda \rho$ is thus a measurable function with respect to the σ-subalgebra $\mathscr{B}_\lambda$ (of $\mathscr{B}$), which contains only the sets that are formed from unions of complete cells of the partition ξ_λ. This being so, the part $P_\lambda \rho$ cannot take different values in the points of any given cell of ξ_λ and hence it cannot distinguish the points on any given cell of ξ_λ. Since ξ_μ is finer than ξ_λ for $\mu > \lambda$, $P_\lambda \rho$ cannot also, *a fortiori*, distinguish the points on the same cell of ξ_μ with $\mu \geqslant \lambda$. Thus, the part of ρ that can possibly distinguish the points on the same cell of any partition ξ_μ with $\mu > \lambda$ is $\rho - P_\lambda \rho$. The dispersion or mean square deviation of $\rho - P_\lambda \rho$ may be taken as an indicator of the degree of difficulty (or, rather, of ease) with which one can achieve the desired distinguishability: *the smaller this dispersion, the greater the degree of difficulty of achieving the desired distinguishability with the given* ρ.

This quantity is easily seen to be

$$\int_\Omega (\rho - P_\lambda \rho)^2(\omega)\, d\mu = \|\rho - P_\lambda \rho\|^2 = \left\| \int_\lambda^\infty dF_\mu \rho \right\|^2$$

(the mean value of $\rho - P_\lambda \rho$ is, of course, zero).

If the evolution of ρ is described by U_t obtained from the deterministic evolution of the phase points, then the degree of difficulty of achieving the desired distinguishability of points belonging to any partition ξ_μ with $\mu \geqslant \lambda$ is

$$\|U_t \rho - P_\lambda U_t \rho\|^2 = \|\rho - U_t^* P_\lambda U_t \rho\|^2 = \|\rho - P_{\lambda - t} \rho\|^2 = \left\| \int_{\lambda - t}^\infty dF_\mu \rho \right\|^2$$

Here the first equality follows from the unitarity of U_t, and the second equality makes use of the imprimitivity property (iii) of P_λ. Since

$$\left\| \int_{\lambda - t}^\infty dF_\mu \rho \right\|^2 \geqslant \left\| \int_\lambda^\infty dF_\mu \rho \right\|^2 \qquad \text{for} \quad t \geqslant 0$$

it follows that if ρ evolves under U_t, then with increasing time it would become progressively *easier* to achieve the desired distinguishability! This conclusion is of course an expression of the fact that for highly unstable systems such as K-flows the deterministic description of motion constitutes an unphysical idealization and it is the dynamics itself that is pointing to its own limitation.

The situation is completely different for the transformed state $\Lambda \rho$ and the entropy-increasing evolution W_t. The indicator of difficulty of distinguishing the points in a cell of ξ_μ with $\mu \geqslant \lambda > 0$ with the state ρ is again

$$\|P_\lambda \Lambda \rho - \Lambda \rho\|^2 = \|\Lambda(P_\lambda \rho - \rho)\|^2 = \left\| \int_\lambda^\infty \frac{1}{1 + e^{K\mu}}\, dF_\mu \rho \right\|^2$$

$$\leqslant \left(\frac{1}{1 + e^{K\lambda}} \right)^2 \left\| \int_{\lambda - t}^\infty dF_\mu \rho \right\|^2 < e^{-2K\lambda} \left\| \int_{\lambda - t}^\infty dF_\mu \rho \right\|^2$$

Since it follows from the above inequality that $\|P_\lambda \Lambda\rho - \Lambda\rho\|^2$ is smaller than the corresponding quantity for ρ by at least the exponential factor $e^{-2K\lambda}$, it is exponentially more difficult to achieve the desired discernability with $\Lambda\rho$ than with ρ.

More importantly, under the evolution W_t to which the transformation Λ leads, the degree of difficulty of achieving the distinguishability of the points belonging to a contracting cell increases progressively with increasing time, in complete contrast to the previously mentioned unphysical behavior of U_t in this regard. This follows from the fact that the semigroup W_t is known to be a so-called *monotonic Markov semigroup*, which satisfies the following condition: $\| W_t \Lambda\rho - 1\|^2$ decreases *strictly monotonically* to zero as $t \to \infty$. It thus follows that

$$\| W_t \Lambda\rho - P_\lambda W_t \Lambda\rho\|^2 = \| W_t \Lambda\rho - 1 + 1 - P_\lambda W_t \Lambda\rho\|^2$$
$$\leqslant \| W_t \Lambda\rho - 1\|^2 + \|1 - P_\lambda W_t \Lambda\rho\|^2$$

Since $P_\lambda 1 = 1$ and $\|P_\lambda\| = 1$, it follows from the above that

$$\| W_t \Lambda\rho - P_\lambda W_t \Lambda\rho\|^2 \leqslant 2 \, \| W_t \Lambda\rho - 1\|^2$$

Since the right-hand side decreases strictly monotonically to zero as $t \to \infty$, the desired assertion of progressively increasing degree of inability to distinguish the points of the contracting fibers follows. It can further be shown that this increase in the degree of *operational indistinguishability* we are discussing is exponential for a large class of states $\Lambda\rho$, corresponding to the fact that for this class of $\Lambda\rho$, the expression $\| W_t \Lambda\rho - 1\|^2$ decreases exponentially fast to zero with increasing time. In fact this class consists of all the states of the form $\Lambda(I - F_\mu)\rho$ with μ any finite number. There are, however, states $\Lambda\rho$ for which the rate of decrease of $\| W_t \Lambda\rho - 1\|^2$, which may be taken as a measure of the rate of approach to equilibrium state 1, is slower than exponential.[(32)]

The preceding discussion should make it clear that the natural requirement that the *physically* realizable states and their physical evolution should be such that with their aid the *operational difficulty* of distinguishing the points on the contracting K-cells should increase progressively with time (for $t \to \infty$) is closely related and consistent with the necessity of the passage from the dynamical evolution U_t to the evolution W_t under which the states approach monotonically to the microcanonical ensemble. As said before, the origin of irreversibility (corresponding to monotonic increase of entropy) is thus directly traced to K-flow instability as expressed by the existence of the K-partition.

The existence of the internal time operator T, apart from its usefulness in serving as the characteristic property of K-flows and in the construction of Λ, is of intrinsic physical interest. In fundamental physical theories, be it classical dynamics, quantum theory, or relativity, time appears only as an external parameter and not as a dynamical variable. For example, it is meaningless to ask in these theories about the (average) age of a given state. This fact is highlighted in the usual formulation of quantum theory, where one cannot get a meaningful answer about the probability that an unstable particle prepared at time $t=0$ will stay undecayed *throughout* an entire time internal, say $[0, t]$, or about the probability that it will decay at *some* point belonging to a given time interval. (This is related to the so-called quantum Zeno effect or paradox.[33,34])

The existence of an internal time operator allows one to associate an (average) age $\langle T\rangle_{\Lambda\rho}$ to states $\Lambda\rho$ given by the formula

$$\langle T\rangle_{\Lambda\rho} = \langle \Lambda\rho, T\Lambda\rho\rangle / \langle \Lambda\rho, \Lambda\rho\rangle$$

The defining condition (a) of internal time just expresses the fact that the average age of states increases under the dynamical evolution U_t in step with the external time parameter t.

The operator T cannot, however, serve as the age operator for the evolution W_t, for the simple reason that $\langle W_t\Lambda\rho, TW_t\Lambda\rho\rangle / \langle W_t\Lambda\rho, W_t\Lambda\rho\rangle$ is not *necessarily* larger than $\langle \Lambda\rho, T\Lambda\rho\rangle / \langle \Lambda\rho, \Lambda\rho\rangle$ for $t \geqslant 0$.

In other words, the average age will not necessarily advance in the same direction as the external time parameter t. The operator $T/\Lambda^2 = T'$ has, however, the required property that under the evolution W_t, the average age defined in terms of T' will increase with the increase t (for $t>0$), although *not linearly with t*. It may be mentioned that if the normalization factor in the definition of average age (the denominator in the definition of $\langle T\rangle_{\Lambda\rho}$ or $\langle T'\rangle_{\Lambda\rho}$) is omitted, then the average age defined with T and T' will increase linearly with t under the evolution U_t and W_t, respectively: the scale of linear increase will, however, depend on the initial state. As a final remark in this connection, I mention that the internal time operator of quantum systems (which can be defined, if at all, only as a non-factorizable operator acting on density operators that does not preserve the purity of states[14] can be shown to satisfy the much discussed time–energy uncertainty relation of quantum mechanics.[35]

3. *K*-FLOW STRUCTURE OF RELATIVISTIC FIELDS

We come now to the main result of this paper: The demonstration of the K-flow structure of the fields. As said before, I shall outline the argument only for a real Klein–Gorden field $\phi(x, t)$ satisfying the equation

$$\frac{\partial^2\phi}{\partial t^2} = \Delta\phi - m^2\phi$$

To this end, we have first to define a suitable phase space for the system and a probability measure μ on the phase space that is invariant under time evolution. The existence of a time operator with the properties mentioned in the last section, or equivalently the existence of an imprimitivity system of conditional expectations in $\mathscr{L}^2_\mu$, will then establish the K-flow property of the field.

For phase space it is natural to take some suitable class of initial Cauchy data or equivalently the associated set of solutions of the field equation, there being a well-known one–one correspondence between initial Cauchy data and the solutions of the field equation. For our purpose it will be more convenient to work with the solutions.

As is well known every real solution $\phi(\mathbf{x}, t) = \phi(x)$ of the KG equation can be written in the form

$$\phi(x) = \int_{k^2 = m^2} e^{ikx} \tilde{\phi}(k) \frac{d^3k}{|k_0|}, \qquad d^3 = dk_1\, dk_2\, dk_3 \tag{1}$$

Here x and k stand for four-vectors $(x_0 = t, x_1, x_2, x_3)$ and (k_0, k_1, k_2, k_3), respectively; $kx = k_0 x_0 - \mathbf{k} \cdot \mathbf{x}$, where $\mathbf{k}$ and $\mathbf{x}$ are the spatial three-vectors of k and x, respectively, and $k^2 = k_0^2 - |\mathbf{k}|^2$. The function $\tilde{\phi}(k)$ on the two hyperboloids defined by the condition $k^2 = m^2$ is required to satisfy the condition

$$\tilde{\phi}(-k_0, -\mathbf{k}) = \phi^*(k_0, \mathbf{k})$$

which guarantees that the solution is real-valued.

If the functions $\tilde{\phi}(k)$ are restricted by the condition

$$\int_{k^2 = m^2} |\tilde{\phi}(k)|^2 \frac{d^3k}{k_0} < \infty$$

then it is well known that the corresponding set of solutions form a real Hilbert space $\mathscr{H}_{\mathrm{R}}$ when the inner product $\langle \phi, \psi \rangle_{\mathrm{R}}$ of the solutions $\phi(x)$ and $\psi(x)$ expressed in terms of the corresponding functions $\tilde{\phi}(k)$ and $\tilde{\psi}(k)$ is given by the formula

$$\langle \phi, \psi \rangle_{\mathrm{R}} = \langle \tilde{\phi}, \tilde{\psi} \rangle_{\mathrm{R}} = \int_{k^2 = m^2} \tilde{\phi}^*(k)\, \tilde{\psi}(k) \frac{d^3k}{|k_0|}$$

This real Hilbert space of solutions can, of course, be described also in terms of the initial Cauchy data corresponding to the solutions. However, the expression of the scalar product $\langle\ ,\ \rangle_{\mathrm{R}}$ is somewhat complicated when expressed in terms of initial Cauchy data and we do not need them here.

What is important for us is the well-known fact that the operators $U_{(a,\Lambda)}$ on $\mathscr{H}_{\mathrm{R}}$ defined by the relation

$$(U_{(a,\Lambda)}\phi)(x) = \phi(\Lambda^{-1}(x-a))$$

provide a *unitary* (orthogonal) representation of the Poincaré group. Here Λ stands for a Lorentz transformation, and a represents four-vector translation. The (anti-self-adjoint) generators of this representation are well known and satisfy the well-known commutation relations of Poincaré–Lie algebra. We need not write them down here.

To fix our notations, let us denote by $U_\tau \equiv e^{P_0\tau}$ the group of time evolution whose action on $\phi(t, \mathbf{x}) \in \mathscr{H}_{\mathrm{R}}$ is given by

$$U_\tau \phi(t, \mathbf{x}) = \phi(t+\tau, \mathbf{x})$$

Thus, the anti-self-adjoint generator P_0 of time evolution is the operator $\partial/\partial t$, and $(P_0)^2 = \partial^2/\partial t^2 = \Delta - m^2 I$ on the space of solutions. Similarly, let P^i $(i=1, 2, 3)$ and N^i be the anti-self-adjoint generator of spatial translation and boost along the x_i direction, respectively; i.e.,

$$\{[\exp(\mathbf{P}\boldsymbol{\alpha})]\phi\}(t, \mathbf{x}) = \phi(t, \mathbf{x} - \boldsymbol{\alpha})$$

$$\{[\exp(\zeta^i N^i)]\phi\}(x) = \phi(\Lambda^{-1}x)$$

where Λ is the boost with velocity $v = \tanh \zeta$ $(c=1)$ along the x_i direction. Then these generators satisfy the commutation relations

$$[P^\mu, P^\nu] = 0, \qquad \mu, \nu = 0, 1, 2, 3$$

$$[N^i, P^0] = -P^i$$

$$[N^i, P^j] = -\delta_{ij} P^0$$

Moreover,

$$(P_0^0)^2 = \sum_{i=1}^{3} (P^i)^2 - m^2 = \Delta^2 - m^2 I$$

It can be easily verified that

$$U_{-t} N^i U_t = e^{-P^0 t} N^i e^{P^0 t} = [e^{-P^0 t}, N^i] + N^i = N^i - tP^i$$

On the other hand, $P^i(P^0)^{-2}$ commutes with U_t. Thus, the operator $N^i(P^i(P^0)^{-2})$ (with summation with respect to repeated i, 1, 2, 3, implied henceforth) satisfies the relation

$$U_{-t} N^i [P^i(P^0)^{-2}]\, U_t = N^i [P^i(P^0)^{-2}] - t\, |\mathbf{P}|^2 (P^0)^{-2}$$

where we have used $|\mathbf{P}|^2$ to denote $\sum_{i=1}^{3}(P^i)^2$. Using the fact that

$$(P^0)^2 = |\mathbf{P}|^2 - m^2 I$$

we can rewrite the above relation to read

$$U_{-t} N^i [P^i (P^0)^{-2}]\, U_t = N^i [P^i (P^0)^{-2}] - tI + t\frac{m^2}{(P^0)^2} \tag{2}$$

Because of the anti-adjointness of the generators [i.e., $(N^i)^\dagger$, etc., where the dagger signifies Hermitian conjugation with respect to the inner product $\langle\ ,\ \rangle_{\mathrm{R}}$], it follows that $(N^i [P^i (P^0)^{-2}])^\dagger = [P^i (P^0)^{-2}]\, N^i$, and, since $U_t^\dagger = U_{-t}$, by taking the Hermitian conjugate of both sides of Eq. (1), we obtain

$$U_{-t} [P^i (P^0)^{-2}]\, N^i U_t = [P^i (P^0)^{-2}]\, N^i - tI + t\frac{m^2}{(P^0)^2}$$

Thus, the self-adjoint operator

$$T'_{\mathrm{R}} = \sum_{i=1}^{3} \frac{N^i P^i (P^0)^{-2} + P^i (P^0)^{-2} N^i}{2}$$

satisfies the relation

$$U_{-t} T'_{\mathrm{R}} U_t = T'_{\mathrm{R}} - t\left(I - \frac{m^2}{(P^0)^2}\right) \tag{3}$$

Actually, the operator T' is shown to be Hermitian only. Using the known domain properties of the generators of the Poincaré group, it can be shown that it is densely defined and it is known that such operators in a *real* Hilbert space always have a self-adjoint extension.[37] By T' we shall mean such a self-adjoint extension.

The relation (3) shows that $T_{\mathrm{R}} \equiv -T'_{\mathrm{R}}$ satisfies condition (a) of the time operator for the evolution U_t of a massless KG field. Moreover, even for a massive field

$$U_{-t} T_{\mathrm{R}} U_t = T_{\mathrm{R}} + t\left(I + \frac{m^2}{-\Delta + m^2}\right)$$

and therefore if the average age of a solution $\phi \in \mathscr{H}_{\mathrm{R}}$ is defined by $\langle \phi, T\phi \rangle_{\mathrm{R}} / \langle \phi, \phi \rangle_{\mathrm{R}}$, it will increase under the evolution U_t linearly with the increase of the external time parameter t, although the scale of this linear increase will depend on ϕ due to the presence of the positive operator

$m^2/(-\Delta+m^2)$. Thus, even for a massive KG field the operator T acts as a generalized time operator, which allows one to associate average to solutions (or equivalently initial Cauchy data) of the field equation.

As a passing remark, let us note that for a *massless* field

$$\left[\sum_{i=1}^{3} N^i P^i (P^0)^{-1}, P^0\right] = \sum_{i=1}^{3} [N^i, P^0]\, P^i (P^0)^{-1} = -P^0$$

This is the same commutation relation as that between the generator D of dilatation or scale transformation and P^0,

$$[D, P^0] = -P^0 \tag{4}$$

Thus, for the massless field the time operator is:

$$T_{\mathrm{R}} = -[D(P^0)^{-1} + (P^0)^{-1} D] \tag{5}$$

The dilatation generator D is, of course, defined as an operator in $\mathscr{H}_{\mathrm{R}}$ even for a massive field and satisfies the commutation relation (4). From this, it can be shown that the operator T_{R} defined by relation (5) will satisfy the relation

$$U_{-t} T_{\mathrm{R}} U_t = e^{-P^0 t} T_{\mathrm{R}} e^{P^0 t} = T_{\mathrm{R}} + tI \tag{6}$$

even for a massive field. However, in this case the operator D is not anti-self-adjoint, corresponding to the fact that the Poincaré invariant measure $d^3k/|k_0|$ on the hyperboloids is not invariant under the action of dilatation group for given fixed mass $m \neq 0$, which is, of course, equivalent to the known fact that the massive Klein–Gordon equation is not invariant under the dilatation group unless one also allows the mass m to change under the transformation. Since D is not anti-self-adjoint, the operator defined through relation (5) is *not* self-adjoint for the massive Klein–Gordon field, although it satisfies the required relation (5) for time operator. In fact, for the massive Klein–Gordon field there *cannot* exist a *self-adjoint* operator T satisfying the required relation (6).

The existence of a self-adjoint operator T satisfying the relation (6) for the massless KG equation is not sufficient by itself to prove the K-flow property. For we have not yet even formulated the defining condition (b) of internal time operator given in the previous section. In other words, we have not defined a probability measure on $\mathscr{H}_{\mathrm{R}}$ which is invariant under the evolution U_t and have not shown that the projections $P_\lambda = F_\lambda + P_{-\infty}$ are projections of conditional expectations, where F_λ are the spectral projections of the operator $\mathbb{T}_{\mathrm{R}}$ that act on functions on $\mathscr{H}_{\mathrm{R}}$ and are induced from the action of $\mathbb{T}_{\mathrm{R}}$ on elements in $\mathscr{H}_{\mathrm{R}}$. As a matter of fact, such a measure cannot exist on $\mathscr{H}_R$. This can be seen from the fact that the internal time operator, when it exists, cannot be a transformation induced from a point transformation mapping the phase space into itself. Let us, however, give an independent proof of the nonexistence of an invariant measure on $\mathscr{H}_R$.

Proposition. There exists no measure μ on the (real) Hilbert space $\mathscr{H}_R$ satisfying the condition $\int \|f\|^2 \, d\mu(f) < \infty$ for every $f \in \mathscr{H}_R$ that is invariant under the evolution group U_t.

The proof follows from a known result (39) that if μ has the property states in the proposition, then for any two given elements g_1 and g_2 in $\mathscr{H}_R$

$$\int_{\mathscr{H}_R} \langle f, g_1 \rangle_R \langle f, g_2 \rangle_R \, d\mu(f) = \langle g_1, A g_2 \rangle_R$$

where A is *necessarily* a Hermitian, nonnegative *nuclear* operator. Thus, if the measure were invariant under U_t, we would have

$$\begin{aligned}
\langle g_1, A g_2 \rangle_R &= \int_{\mathscr{H}_R} \langle f, g_1 \rangle_R \langle f, g_2 \rangle_R \, d\mu(U_t f) \\
&= \int_{\mathscr{H}_R} \langle U_{-t} f, g_1 \rangle_R \langle U_{-t} f, g_2 \rangle_R \, d\mu(f) \\
&= \int_{\mathscr{H}_R} \langle f, U_t g_1 \rangle_R \langle f, U_t g_2 \rangle_R \, d\mu(f) \\
&= \langle U_t g_1, A U_t g_2 \rangle_R = \langle g_1, U_{-t} A U_t g_2 \rangle_R
\end{aligned}$$

for any $g_1, g_2 \in \mathscr{H}_R$.

This is possible only if U_t commutes with A and hence also with the projections to its eigenspaces. Since A is nuclear, it has (at least one) finite-dimensional eigenspace. The unitary operator U_t, on the other hand, has absolutely continuous spectrum and hence it cannot commute with finite-dimensional projections. It may noted that all Gaussian measures on $\mathscr{H}_R$ satisfy the condition on μ stated in the proposition. Thus, in particular, there is no invariant Gaussian measure on $\mathscr{H}_R$.

These considerations, then, show that in order to obtain an invariant measure and to proceed further in demonstrating the K-flow structure of fields, we have to consider a larger space than the space $\mathscr{H}_R$ as phase space.

Before discussing this, it should be mentioned that the existence of the time operator T_R in $\mathscr{H}_R$ for the wave equation is equivalent to the existence of the so-called outgoing and incomming subspaces, which play important roles in the beautiful theory of scattering developed by Lax and Phillips.[38]

In fact, if E_λ represents the spectral projections of T_R and D_- the range of E_0, then D_- has the properties of the *incoming* subspace of the Lax–Phillips theory. Conversely, if an incoming subspace exists and F_0 denotes the projection onto it, then the projections F_λ defined by $F_\lambda = U_\lambda F_0 F_{-\lambda}$ (where U_λ is the time evolution unitary group) constitute the spectral projection of a time operator T that satisfies $U_{-\lambda} T U_\lambda = T + \lambda I$. The existence of T in $\mathscr{H}_R$ therefore provides a group-theoretic proof of the existence of incoming and outgoing subspaces for a (free) wave equation, a fact first recognized by Lax and Phillips and forming the starting point of their theory of scattering.

There is, however, the difference that the incoming and outgoing subspaces corresponding the operator T_R considered here are subspaces in the Hilbert space $\mathscr{H}_R$ with relativistically invariant inner product, whereas Lax and Phillips work in the Hilbert space (of Cauchy data) defined by the so-called energy norm, which is not relativistically invariant. On the other hand, the Lax–Phillips incoming and outgoing subspaces D_- and D_+, and hence also the spectral projections of the time operator corresponding to them, can be described very simply: D_- consists of all solutions (with finite energy norm) of the wave equation that *vanish* inside the backward light cone of a given space-time point (the here and now of an observer) and D_+ consists of those solutions that vanish inside the forward light cone of the given spacetime point. Such a simple description does not hold for the spectral projections of T_R constructed here. The time operator T_R given here in terms of the generators of the Poincaré group and the time operator constructed from the incoming subspace D_- of Lax and Phillips are, however, connected by a transformation that allows one to determine the spectral projections of T_R in terms of the projection onto the incoming (or outgoing) subspace considered by Lax and Phillips. This relationship will be discussed in a subsequent publication.

Coming back to the appropriate enlargement of the phase space that will permit the definition of an invariant measure, we shall consider the space of distribution (in the sense of generalized function)-valued solutions of the KG equation. More precisely, we shall allow the function $\tilde{\psi}(k)$ [associated with a solution $\psi(x)$ through formula (1)] to be in the dual $\mathscr{S}'$ of the space $\mathscr{S}$ of rapidly decreasing Swartz test functions on the hyperboloids $k^2 = m^2$. If a locally integrable function of polynomial growth $\tilde{\psi}(k)$ is to be considered as a distribution, the value $\langle \psi, \phi \rangle$ of the corresponding linear functional for a test function $\tilde{\phi}(k) \in \mathscr{S}$ is defined to be

$$\langle \psi, \phi \rangle \equiv \int_{k^2 = m^2} \tilde{\psi}^*(k)\, \tilde{\phi}(k)\, d^3k = \left[\int_{k^2 = m^2} \tilde{\psi}(k)\, \phi^*(k)\, d^3k \right]^*$$

But the $\mathscr{S}'$ is, of course, larger than the space of locally integrable functions. From the relation (1) between solutions $\phi(x)$ and the corresponding function $\tilde{\phi}(k)$ on the hyperboloids, it clear that the action of the time evolution operator U_τ on $\tilde{\phi}(k)$ is just that of multiplication operator by $e^{ik_0\tau}$,

$$U_\tau \tilde{\phi}(k) = e^{ik_0\tau}\tilde{\phi}(k)$$

This is also the action of evolution group U_τ for distribution-valued solutions. Therefore, if the linear functional corresponding to given distribution-valued solution ψ assigns to the test function ϕ the value $\langle \psi, \phi \rangle$, the time-evolved solution $U_\tau \psi$ will assign to ϕ the value

$$\langle U_\tau \psi, \phi \rangle = \langle \psi, U_{-\tau}\phi \rangle$$

Now,

$$B(\tilde{\phi}_1, \tilde{\phi}_2) \equiv \int_{k^2 = m^2} \tilde{\phi}_1^*(k)\, \tilde{\phi}_2(k) \frac{d^3k}{|k_0|} \equiv \langle \phi_1 \phi_2 \rangle_{\mathrm{R}}$$

the relativistic scalar product, defines a positive-definite quadratic form on $\mathscr{S}$ and there exists a Gaussian measure μ on $\mathscr{S}'$ whose characteristic functional

$$C(\alpha\phi) \equiv \int_{\mathscr{S}'} e^{i\langle \psi, \alpha\phi \rangle}\, d\mu(\psi)$$

(for every given $\phi \in \mathscr{S}$) is given by

$$C(\alpha\phi) = e^{-\alpha^2 \langle \phi, \phi \rangle_{\mathrm{R}}} \tag{7}$$

[This follows from Minlos' theorem[40]; we have, however, glossed over the technical but important point that $B(\tilde{\phi}_1, \tilde{\phi}_2)$ has to be shown to be weakly continuous in $\mathscr{S}$.] The invariance of this measure under evolution U_t follows immediately because

$$\begin{aligned}\int_{\mathscr{S}'} e^{i\langle \psi, \phi \rangle_{\mathrm{R}}}\, d\mu(U_\tau \psi) &= \int_{\mathscr{S}'} e^{i\langle U_{-\tau}\psi, \phi \rangle_{\mathrm{R}}}\, d\mu(\psi) \\ &= \int_{\mathscr{S}'} e^{i\langle \psi, U_\tau \phi \rangle_{\mathrm{R}}}\, d\mu(\psi) \\ &= e^{-\langle U_\tau \phi, U_\tau \psi \rangle_{\mathrm{R}}} \\ &= e^{-\langle \phi, \phi \rangle_{\mathrm{R}}}\end{aligned}$$

which shows that the transformed measure under U_t has the same characteristic functional as the untransformed measure.

To sum up, the phase space for the KG field is taken to be the space of distribution-valued solutions ψ,

$$\psi(x) = \int_{k^2 = m^2} e^{ikx} \tilde{\psi}(k) \frac{d^3k}{|k_0|}$$

with $\tilde{\psi}(k) \in \mathscr{S}'$, where $\mathscr{S}$ is the space of rapidly decreasing Swartz test functions $\tilde{\phi}(k)$ in $\mathscr{H}_{\mathrm{R}}$. The invariant measure μ on $\mathscr{S}'$ is the Gaussian measure whose characteristic functional is given by (7). Every given test function in $\tilde{\phi}(k)$ (or the corresponding solution) defines a function $\phi(\psi)$ on $\mathscr{S}'$: $\psi \to \langle \psi, \phi \rangle$, the value of the linear functional corresponding to ψ assumes on ϕ. In fact, since $\mathscr{S}$ is dense in $\mathscr{H}_{\mathrm{R}}$, every element in $\mathscr{H}_{\mathrm{R}}$ defines a function on $\mathscr{S}'$. These functions on measure space $(\mathscr{S}', \mu)$ are Gaussian random variables with mean zero and variance

$$\langle \phi_1(\psi), \phi_2(\psi) \rangle \equiv \int_{\mathscr{S}'} \phi_1(\psi)\, \phi_2(\psi)\, d\mu = \langle \phi_1, \phi_2 \rangle_{\mathrm{R}}$$

Let us also mention that the σ-algebra $\mathscr{B}$ of measurable subsets for μ is the σ-algebra generated by the so-called cylindrical sets on $\mathscr{S}'$, which we need not describe here.[40] Moreover, $\mathscr{B}$ is the *smallest* σ-algebra with respect to which all functions $\phi(\psi)$ on $\mathscr{S}'$ with $\phi \in \mathscr{H}_{\mathrm{R}}$ are measurable.

One can now demonstrate the K-flow flow structure of the massless field as follows. Let E_λ be the spectral projection of the time operator T and D_λ the subspace corresponding to E_λ. It is clear that

$$U_t D_\lambda = D_{\lambda + t} \supset D_\lambda \qquad \text{for} \quad t > 0$$

Now, the function $\phi(\psi)$ on $\mathscr{S}'$ that a given $\phi \in \mathscr{H}_{\mathrm{R}}$ defines satisfies

$$\phi(U_t \psi) = (U_{-t} \phi)(\psi)$$

or

$$\phi_t(\psi_t) = \phi(\psi)$$

where we have written ϕ_t to denote the time-evolved solution $U_t \phi$ and similarly

$$\psi_t = U_t \psi$$

Thus, if $\mathscr{A}$ is the smallest σ-algebra of sets in $\mathscr{S}'$ with respect to which the functions defined by a given set D of test functions are measurable, then $U_t \mathscr{A}$ is the smallest σ-algebra of sets in $\mathscr{B}$ with respect to which the functions defined by test functions in the set $U_t D$ are measurable. Let us denote by $\mathscr{B}_0$ the smallest σ-subalgebra of $\mathscr{B}$ with respect to which functions on $\mathscr{S}'$ defined by ϕ belonging to D_0 (the subspace of the spectral projection E_0 of T) are measurable. It then follows that $\mathscr{B}_\lambda \equiv U_\lambda \mathscr{B}_0$ is the smallest σ-algebra with respect to which functions defined by $\phi \in D_\lambda \equiv U_\lambda D_0$ are measurable. Since $D_\lambda > 0$, it follows that $\mathscr{B}_\lambda \supset \mathscr{B}_0$ for $\lambda > 0$. In fact, since D_λ is an increasing family of subspaces of $\mathscr{H}_R$ with

$$\bigvee_{-\infty < \lambda < \infty} D_\lambda = \mathscr{H}_R \qquad \text{and} \qquad \bigwedge_{-\infty < \lambda < \infty} D_\lambda = \{0\}$$

it follows that $\mathscr{B}_\lambda \supset \mathscr{B}_\mu$, if $\lambda > \mu$, $\bigcup_{-\infty < \lambda < \infty} \mathscr{B}_\lambda = \mathscr{B}$, and $\bigcap_{-\infty < \lambda < \infty} \mathscr{B}_\lambda =$ the trivial σ-algebra that consists of sets of either measure 0 or 1. This estabishes the K-flow structure of massless KG field.

The preceding discussion could have been carried out in terms of the existence of a K-partition. In fact, the σ-algebra $\mathscr{B}_0$ will define a partition ξ_0 of the phase space into disjoint cells such that every set in $\mathscr{B}_0$ will be the union of some of the complete cells of this partition. This partition has all the properties of the K-partition. All points (i.e., distribution-valued solutions) belonging to a single cell of ξ_0 have the property that they all assign the same value to every test function ϕ belonging to D_0.

The partition corresponding to $U_\lambda \mathscr{B}_0 = \mathscr{B}_\lambda$ will be finer (for $\lambda > 0$) as $\mathscr{B}_\lambda \supset \mathscr{B}_0$. In fact, all points belonging to a single cell of $\mathscr{B}_\lambda$ will assign the same value to every test function ϕ in $U_\lambda D_0 = D_\lambda \supset D_0$. This shows that the degree of difficulty in distinguishing the point belonging to a given single cell of the K-partition increases progressively with increasing time t, since they agree (i.e., assign the same value to every ϕ) in progressively increasing subspaces $U_t D$ of test functions.

As for the massive Klein–Gorden field, there is no self-adjoint operator T satisfying

$$U_t^* T U_t = T + tI \qquad \text{for all real } t$$

However, since the evolution operators U_t in $\mathscr{H}_R$ have Lebesgue homogeneous spectrum, one can show that for a discrete subgroup of U_t, e.g., the subgroup U_n, $n = 0, \pm 1, \pm 2,\ldots$, there exist subspaces D_n in $\mathscr{H}_R$ such that:

(i) $U_m D_n \equiv D_{n+m}$

(ii) $D_n \supset D_m$ for $n > m$

(iii) $\bigwedge_{-\infty < n < \infty} D_m \equiv \{0\}$

(iv) $\bigvee_{-\infty < n < \infty} D_n = \mathscr{H}_R$

Thus, by using arguments similar to the one given above, one can show that the discrete cascade U_n of the flow U_t on $\mathscr{S}'$ is a K-system or K-cascade. It is known, however, that if a discrete cascade of a flow is a K-system, then the entire flow is a K-flow.

Instead of the above discussion in terms of σ-algebras, we could have directly proved the K-flow property of the field by showing the existence of an imprimitivity system of *conditional expectations* with respect to the unitary group U_t on $\mathscr{L}^2_\mu$ (μ is the invariant Gaussian measure on $\mathscr{S}'$) induced from the action of U_t on $\mathscr{S}'$,

$$U_t\rho(\psi)=\rho(U_{-t}\psi), \qquad \rho\in\mathscr{L}^2_\mu, \quad \psi\in\mathscr{S}'$$

(see the previous section in this connection).

In fact, such an imprimitivity system can be obtained from the spectral projections E_λ of T. We *cannot*, however, define P_λ simply by first defining $(P_\lambda\phi)(\psi)=(E_\lambda\phi)(\psi)$ and defining it for products $\phi_1(\psi)\,\phi_2(\psi)\cdots$ by

$$P_\lambda[\phi_1(\psi)\,\phi_2(\psi)\cdots]=(E_\lambda\phi_i)(\psi)(E_\lambda\phi_i)(\psi)\cdots$$

This is because the operator P_λ thus defined will not be a projection of conditional expectation. This can be easily seen because the integral (I) $\int_{\mathscr{S}'} P_\lambda[\phi_1(\psi)\,\phi_2(\psi)]\,d\mu$ would be equal to $\int_{\mathscr{S}'}(E_\lambda\phi_1)(\psi)(E_\lambda\phi_2)(\psi)\,d\mu$ which is equal to $\langle E_\lambda\phi_1,\ E_\lambda\phi_2\rangle_R$ and is different from $\langle\phi_1,\ \phi_2\rangle_R$. But if P_λ were to be a conditional expectation the integral (I) should be $\int_{\mathscr{S}'}[\phi_1(\psi)\,\phi_2(\psi)]\,d\mu$ which equals $\langle\phi_1,\phi_2\rangle_R$. However, a slight modification yields the desired result.

Let $\phi_i(\psi)$ be functions on $\mathscr{S}'$, defined by test functions $\phi_i\in\mathscr{S}$ because of the previous consideration we define the action of P_λ on functions $\rho(\psi)$ on $\mathscr{S}'$ not by $(P_\lambda\rho)(\psi)=\rho(E_\lambda\psi)$ where $(E_\lambda\psi,\phi)=(\psi,E_\lambda\phi)$, but through its action on the Ito–Wick polynomial which form a dense set on $\alpha^2_{\mathscr{S}',\mu}$. Explicitly we define the action of P_λ as follows

$$P_\lambda :\phi_{i_1}(\psi)\cdots\phi_{i_n}(\psi):\,=\,:E_\lambda\phi_{i_1}(\psi)\cdots E_\lambda\phi_{i_n}(\psi):$$

Here the symbol $:\cdots:$ denotes the so-called Itô–Wick product of Gaussian random variable $\phi_i(\psi)$. For more details see Ref. 41. It now follows that P_λ thus defined are self-adjoint projections in $\mathscr{L}^2_\mu$ corresponding to conditional expectations. This follows from the fact that P_λ thus defined maps the unit function $1\in\mathscr{L}^2_\mu$ to 1 and is positivity-preserving.[(41)] From the definition of U_t and P_λ and the fact that E_λ form a system of imprimitivity for U_t, one can verify that P_λ are an imprimitivity system for U_t.

To conclude this section, it will be interesting to indicate how the K-flow property of the massless KG equation will go in the setting of the Lax–Phillips theory of incoming and outgoing subspaces. Corresponding to the Hilbert space $\mathscr{H}_R$ we consider now the Hilbert space $\mathscr{H}_E$ of (real) initial Cauchy data

$$f=\begin{pmatrix}\phi\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\phi\\ \pi\end{pmatrix}$$

with the inner product corresponding to the so-called energy from $\|f\|_E$, which is given by

$$\|f\|_E = \int d^3x\, \tfrac{1}{2}(\pi^2 + |\nabla\phi|^2) = \int d^3x\, \tfrac{1}{2}[\pi^2(\mathbf{x}) - \phi(\mathbf{x})(\Delta\phi)(\mathbf{x})]$$

The time evolution U_t of Cauchy data is, as is well known, described by the operator-valued matrix

$$U_t = \begin{pmatrix} \cos Bt & B^{-1}\sin Bt \\ -B\sin Bt & \cos Bt \end{pmatrix}$$

where $B = (-\Delta)^{1/2}$. This means that if

$$\begin{pmatrix} \phi(\mathbf{x}) \\ \pi(\mathbf{x}) \end{pmatrix}$$

is the initial Cauchy data, then

$$\phi(\mathbf{x}, t) = \cos Bt\phi + B^{-1}\sin Bt\pi$$

is the solution of the wave equation with $\phi(\mathbf{x}, t=0) = \phi(\mathbf{x})$ and

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t)\bigg|_{t=0} = \pi(\mathbf{x})$$

It is known that U_t defined as above is unitary with respect to the scalar product corresponding to the energy norm.

Furthermore, in the Hilbert space $\mathscr{H}_E$ there exists a self-adjoint time operator T whose spectral projections P_λ are given by $U_\lambda P_0 U_\lambda^*$, where P_0 is the projection onto the subspace of all (initial) real Cauchy data with finite norm such that the solution corresponding to them vanishes in the backward light cone of a fixed space-time point. In other words, P_0 is the projection onto the real subspace of the incoming subspace D_- considered byLax and Phillips. We need not give here the explicit form of T, as it will be given elsewhere. For the same reasons as before one cannot find a measure on $\mathscr{H}_E$ that is invariant under time evolution.

In order to define an invariant measure, we have to consider, as before, a larger space as the phase space of the system. This larger space is the set of all Swartz distribution-valued Cauchy data. In other words, the phase space is

$$\mathscr{S}'_{(\mathbb{R}^3)} \oplus \mathscr{S}'_{(\mathbb{R}^3)}$$

Consider the bilinear form

$$B(f, g), \qquad f = \begin{pmatrix} \phi \\ \pi \end{pmatrix}, \quad g = \begin{pmatrix} \phi' \\ \pi' \end{pmatrix}$$

given by

$$B(f, g) = \int d^3x \, [-\phi(\mathbf{x}) \, \Delta^{-1} \phi'(\mathbf{x}) + \pi(\mathbf{x}) \, \pi'(\mathbf{x})]$$

on Swartz test function-valued initial data; i.e., on

$$\mathscr{S}_{(\mathbb{R}^3)} \oplus \mathscr{S}_{(\mathbb{R}^3)}$$

Now, as before, we can define a measure μ on

$$\mathscr{S}'_{(\mathbb{R}^3)} \oplus \mathscr{S}'_{(\mathbb{R}^3)}$$

whose characteristic functional

$$C(\alpha f), \qquad f \in \mathscr{S}_{(\mathbb{R}^3)} \oplus \mathscr{S}_{(\mathbb{R}^3)}$$

is given by

$$C(\alpha f) = \int_{\mathscr{S}'_{\mathbb{R}^3} \oplus \mathscr{S}'_{\mathbb{R}^3}} e^{is\alpha(f)} \, d\mu(s) = e^{-\alpha^2 B(f,f)}$$

Here s denotes a Swartz distribution-valued Cauchy data, i.e., an element, say

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathscr{S}'_{(\mathbb{R}^3)} \oplus \mathscr{S}'_{(\mathbb{R}^3)}$$

and for

$$f = \left(\begin{pmatrix} \phi \\ \pi \end{pmatrix}\right) \in \mathscr{S}_{(\mathbb{R}^3)} \oplus \mathscr{S}_{(\mathbb{R}^3)}$$

$s(f)$ denotes $s_1(\phi) + s_2(\pi)$.

This measure can be shown to be invariant under time evolution of distribution-valued (initial) Cauchy data in accordance with the wave equation. We need not stop here to prove this.

The proof of the K-flow property of the wave equation can now proceed as before. In fact, if $\mathscr{B}_0$ is the smallest σ-algebra of subsets of

$$\mathscr{S}'_{(\mathbb{R}^3)} \oplus \mathscr{S}'_{(\mathbb{R}^3)}$$

with respect to which the function $s(f) = f(s)$ defined on

$$\mathscr{S}'_{(\mathbb{R}^3)} \oplus \mathscr{S}'_{(\mathbb{R}^3)}$$

with f of the form $\binom{-\Delta\phi}{\pi}$, where $\binom{\phi}{\pi}$ are in D_- and ξ_0 is the corresponding partition, then it can be shown as before that ξ_0 is a K-partition for the time evolution of $\mathscr{S}'_{(\mathbb{R}^3)}$-valued Cauchy data. Details of these will be given elsewhere. Some implications of the results of this section will be briefly discussed in the next section.

4. CONCLUDING REMARKS

The fact that the KG field and even the wave equation have K-flow structure seems to be a rather unexpected result. It is true, of course, that this K-flow structure is visible only if one considers distribution-valued solutions or Cauchy data. At first sight, one might be tempted to argue that distribution-valued solutions or data do not correspond to physically realizable situations, and, as such, the K-flow structure of the field is devoid of physical interest. However, the mathematical analysis of many classical problem, e.g., in fluid dynamics, concerning propagation of singularities does involve consideration of distribution-valued data. Several physical situations, such as flashing on an extremely intense source of light for a very short duration, could be idealized to correspond to distribution-valued initial data. In view of such considerations, it seems to me that the demonstrated K-flow structure of the fields cannot be dismissed as bereft of physical interest.

The K-flow structure of the fields implies instability of their evolution. At first sight, this conclusion seems to be in contradiction with the result concerning the dependence of solutions on initial data, say, for the wave equation.[42] The result alluded to here, however, says, roughly speaking, that given a *prescribed* neighborhood $\mathcal{O}$ of solution and given *a time* t, there exists a neighborhood of $N_{\mathcal{O}}(t)$ of initial data such that solutions corresponding to the data in $N_{\mathcal{O}}(t)$ will be in the neighborhood *for time* t. The important point is that the neighborhood $N_{\mathcal{O}}(t)$ *depends on* t and this dependence could be very irregular and for $t \to \infty$ the neighborhood $N_{\mathcal{O}}(t)$ could shrink to a point. The dependence of $N_{\mathcal{O}}(t)$ on t is, as far as I know, not studied in the existing literature. Thus, the result on the dependence of solutions on initial Cauchy data alluded to here is not in contradiction with the instability of evolution implied by the K-flow structure of the field. In fact, in view of the K-flow structure of fields established here, one can say, paraphrasing J. Lightbill, that there is a fundamental limitation to the predictibility and deterministic description of evolution even in the case of the good old wave equation. The implications of this limitation are the same as discussed in the first part of this article.

As mentioned before, relativistic fields sem to provide the first examples of relativistic systems that are K-flow. One can thus study the relativistic transformation properties of the internal time operator and other objects appearing in our theory of irreversibility. The implications will be the subject of subsequent publications.

Let us conclude on a general note. Our attempt to introduce the principle of irreversibility at the fundamental dynamical level has led us to introduce objects such as the internal time operator T, the transformation Λ constructed from it, and the semigroup W_t of evolution, none of which are like the usual dynamical variables. In particular, we have stressed that these objects introduce certain nonlocal feature into the theory.[10] They also do not preserve the algebraic structure of usual dynamical variables. (The nonlocal features that the principle of irreversibility introduces into the theory are exhibited even more explicitly by the consideration given here. Note in this connection that the time operator T for the wave equation given in the previous section involves the operator $P_0^{-2} = -\Delta^{-2}$, which is known to be a nonlocal operator.) Thus, the introduction of the principle of irreversibility at the fundamental level of dynamics (be it classical, quantum, or relativistic) implies a fundamental modification and enlargement of the conceptual as well as mathematical structure of these theories. To uncover the implications of these modifications is part of the continuing search for a deeper understanding of the enigma of time, and this search, it seems, still holds promises of as yet undreamt of surprises.

ACKNOWLEDGMENT

It is a pleasure to thank Dr. I. Antoniou for numerous discussions as well as for his help in the preparation of this article. I would also like to express my gratitude to Prof. G. Nicolis for his patient encouragement to write this article, and to M. Kies for her patience in deciphering and typing a practically illegible manuscript.

REFERENCES

1. J. Lighthill, *Proc. R. Soc. Lond. A* **407**:35–50 (1986).
2. R. P. Feynman, *The Feynman Lectures on Physics*, Vol. III (Addison-Wesley, Reading, Massachusetts, 1965).
3. B. Misra, *Proc. Natl. Acad. Sci. USA* **75**:1627–1631 (1978).
4. B. Misra, I. Prigogine, and M. Courbage, *Physica* **98A**:1–26 (1979).
5. B. Misra, in *Quantum Mechanics in Mathematics, Chemistry and Physics*, K. Gustafson and W. Reinhardt, eds. (Plenum Press, 1981), pp. 495–502.
6. B. Misra and I. Prigogine, *Prog. Theor. Phys. (Suppl.)* **69**:101–110 (1980).
7. S. Goldstein, B. Misra, and M. Courbage, *J. Stat. Phys.* **25**:111–125 (1981).
8. B. Misra and I. Prigogine, in *Long-Time Prediction in Dynamics*, C. Horton, Jr., L. Reichl, and V. Szebehely, eds. (Wiley, New York, 1983), pp. 21–43.
9. I. Prigogine and M. Courbage, *Proc. Natl. Acad. Sci. USA* **80**:2412–2416 (1983).
10. B. Misra and I. Prigogine, *Lett. Math. Phys.* **7**:421–429 (1983).

11. Y. Elskens and I. Prigogine, *Proc. Natl. Acad. Sci. USA* **83**:5756–5760 (1986).
12. R. Goodrich, K. Gustafson, and B. Misra, *Physica A* **102**:379–388 (1980).
13. R. Goodrich, K. Gustafson, and B. Misra, *J. Stat. Phys.* **43**:317–320 (1986).
14. B. Misra, I. Prigogine, and M. Courbage, *Proc. Natl. Acad. Sci. USA* **76**:4768–4772 (1979).
15. C. Lockhart and B. Misra, *Physica* **136A**:47–76 (1986).
16. M. Spohn, *Rev. Mod. Phys.* **53**:569–615.
17. I. Prigogine, C. George, F. Hénin, and L. Rosenfeld, *Chem. Scr.* **4**:5–32 (1973).
18. C. Lockhart, B. Misra, and I. Prigogine, *Phys. Rev. D* **25**:921–929 (1982).
19. Y. Sinaï, *Sov. Math. Dokl.* **4**:1818–1822 (1963).
20. Y. Sinaï, *Uspekhi Mat. Nauk.* **27**:137 (1972).
21. G. Gallavotti and D. Ornstein, *Commun. Math. Phys.* **38**:83–101 (1974).
22. U. M. Titulaer, *Physica* **70**:456–476 (1973).
23. O. Lanford and J. Lebowitz, in *Lecture Notes in Physics*, No. 38 (1975), p. 144.
24. J. L. Van Hemmen, *Phys. Rep.* **65**(2):43–149 (1980).
25. D. Anosov, *Proc. Steklov Inst.*, No. 90 (1967).
26. Y. Sinaï, *Sov. Math. Dokl.* **1**:335 (1960).
27. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, 1968).
28. I. Cornfeld, S. Fomin, and Y. Sinaï, *Ergodic Theory* (Springer, 1982).
29. R. Bahadur, *Proc. An. Math. Soc.* **6**:565–570 (1955).
30. J. Bell, *Helv. Phys. Acta* **45**:237 (1972).
31. S. Goldstein, *Isr. J. Math.* **38**:241–256 (1981).
32. R. de La Leave, *J. Stat. Phys.* **29**:17 (1982).
33. B. Misra, in *High Energy Physics Symposium* (Bhubaneswar, India, 1976).
34. B. Misra and E. C. G. Sudarshan, *J. Math. Phys.* **18**:756–763 (1976).
35. M. Courbage, *Lett. Math. Phys.* **4**:425–432 (1980).
36. I. E. Segal, *Mathematical Problems of Relativistic Physics* (1963).
37. J. Weidman, *Linear Operation in Hilbert Space* (Springer, 1980).
38. P. Lax and R. Phillips, *Scattering Theory* (Academic Press, 1967).
39. A. Skorohod, *Integration in Hilbert Space* (Springer, 1974).
40. I. Gelfand and N. Vilenkin, *Generalized Functions* (Vol. 4) (Academic Press, 1964).
41. B. Simon, *The* $P(\phi)_2$ *Euclidean (Quantum) Field Theory* (Princeton University Press, 1974).
42. F. Treves, *Basic Linear Partial Differential Equations* (Academic Press, 1975), p. 108, Theorem 135.